Existence, uniqueness and approximation of stochastic Schrodinger equation: the diffusive case

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Abstract

Recent developments in quantum physics make heavy use of so-called "quantum trajectories". Mathematically, this theory gives rise to "stochastic Schrödinger equations", that is, pertubations of Schrödinger-type equations under the form of stochastic differential equations. But such equations are in general not of the usual type as considered in the litterature. They pose a serious problem in terms of: justifying the existence and uniqueness of a solution, justifying the physical pertinence of the equations.

In this article we concentrate on a particular case: the diffusive case, for a two-level system. We prove existence and uniqueness of the associated stochastic Schrödinger equation. We physically justify the equations by proving that they are continuous time limit of a concrete physical procedure for obtainig quantum trajectory.

Introduction

Belavkin equations (also called stochastic Schrödinger equations) are classical stochastic differential equations describing the evolution of an open quantum system undergoing a continuous quantum measurement. The solutions of such equations are called quantum trajectories and describe the time evolution of the state of the system. The random nature of the result of quantum measurement is at the origin of the stochastic character of the evolution.

The first rigorous description of a state undergoing a continuous measurement is due to Davies in [5]. It describes in quantum optics, the behaviour of an atom from which we observe the photon emission. This is the so-called Resonnance Fluorescence experiment (see [9] and [4]) .

In the literature, essentially two kinds of Belavkin equations are considered: they are either driven by a Brownian motion or by a counting process. But the kind of equations which are obtained this way are of non-usual type compared to the usual theory of stochastic differential equations. In particular there is no reference in physics nor in mathematics, where the existence and the uniqueness of the solution of such equations is discussed. Furthemore, the physical justification of the apparition of these equations requires in general quite heavy mathematical framework (Von-Neumann algebra, conditional expectation, filtering...). the high technology of such tools contrasts with the simplicity and the intuition of the physical model.

An approach to such equations, which is physically very intuitive, is the one of repeated quantum interactions. The setup is the following. The continuous measurement model is obtained as a limit of discrete models. This discrete model is a naive approach to the interaction of a simple system interacting with a field. The field is represented as a chain of independent copies of small pieces of environment. The simple system interacts, for a time interval h, with one piece of the environment. After that interaction an observable of the piece of environment is measured. The random result of the measurement induces a random new state for the small system. The small system then interacts again with another piece of the environment for a time interval h. A measurement of the same observable of this second copy is performed. And so on.

This experiment gives rise to a discrete evolution of the state of the small system, which is a Markov chain. The continuous time limit ($h \to 0$) of this evolution should give rise to the quantum trajectories.

Repeated quantum interactions have been considered by Attal-Pautrat in [3] and by Gough in [6]. The continuous limit of repeated quantum interactions is rigorously shown to converge to a quantum stochastic differential equation in [3]. The setup of measuring an observable of the chain after each interaction is considered in [6], but the continuous limit, the existence and the uniqueness of the solutions are not all treated rigorously in this reference.

The aim of this article is to study the diffusive Belavkin equation, to show existence and uniqueness of the solution, to show its approximation by repeated quantum interaction models. The same results for the equation concerning the counting process are developed in another article [14].

This article is structured as follow:

In section (1) we present the model of quantum measurement interaction. In particular we precise the mathematical way to construct the principle of quantum repeated interactions. We define then the sequence of states resulting to the measurement and we show the Markov property. To finish we establish a stochastic finite difference equation which appears as a discrete stochastic differential equation.

The section (2) is devoted to the continuous stochastic Belavkin equation. We deal with the problems of existence and uniqueness of solutions of such equations.

In section (3) we combine the two previous section. Thanks to the discrete model of section (1) we obtain an approximation sequence of states. We define the candidate to be the good approximation of the diffusive solution. To conclude we present the convergence theorem which justifies the diffusive model of Belavkin equation.

1 The discrete quantum trajectory

The main goal of this section is to describe the mathematical model of repeated quantum measurement. We present the general theory of indirect measurement which is necessary to obtain significant information. We describe the evolution of the small system undergoing successive measurement through the discrete quantum trajectory. We focus on probabilistic properties of the sequence representing the discrete quantum trajectory, in particular the Markov character.

1.1 Repeated quantum measurement

This section is devoted to lay out the mathematical framework of the naive physical model presented in the introduction.

To obtain information about the evolution of the state of a system, in quantum optics, the principle of indirect measurement is used. A small system interacts with a field (a photon stream for example) on which we perform a measurement and we are interested in the evolution of the small system. We are going to point out this method in the case of quantum repeated interaction.

The field is represented as a chain of independent copies of pieces of environment. Each copy is represented by a Hilbert space $\mathcal{H}$ and the small system is described by $\mathcal{H}_0$. The copies interact with the small system one after the other. The mathematical description of one interaction is the following.

The compound system is described by the tensor product $\mathcal{H}_0 \otimes \mathcal{H}$ and the interaction is characterized by a unitary evolution U acting on this Hilbert space. In the Schrödinger picture, if ρ denotes any state on the tensor product the evolution is given by:

$$\rho \to U\,\rho\,U^{\star}.$$

Here the time of interaction is not specified. It will be important when considering approximations, all these questions are treated in [3] and this is the subject of the section 3. For instance we consider that this time is fixed (equal to one for example) but we keep in mind that the unitary operator U depends on the interaction length time. After the above interaction, we consider a second copy of $\mathcal{H}$ which interacts with $\mathcal{H}_0$ in the same way.

This sequence of interaction is described by the state space:

$$\mathbf{\Gamma} = \mathcal{H}_0 \otimes \bigotimes_{k\geq 1} \mathcal{H}_k \tag{1}$$

The countable tensor product $\bigotimes_{k\geq 1}\mathcal{H}_k$ means the following. We consider that $\mathcal{H}_0$ and $\mathcal{H}$ are finite dimensional Hilbert spaces. Let $\{X_0, X_1, \dots, X_n\}$ be a fixed orthonormal basis of $\mathcal{H}$, the projector on X_0: $|X_0\rangle\langle X_0|$ being the ground state (or vacuum state) of $\mathcal{H}$ (this is the bra-ket notation in mathematical physics see the remark below). The tensor product is taken with respect to X_0 (for all details concerning countable tensor product see [3]).

Remark: A vector Y in a Hilbert space $\mathcal{H}$ is represented by the application $|Y\rangle$ from $\mathbb{C}$ to $\mathcal{H}$ which acts with the following way $|Y\rangle(\lambda) = |\lambda Y\rangle$. The linear form on $\mathcal{H}$ are represented by the operators $\langle Z|$ which acts on the vector $|Y\rangle$ by $\langle Z||Y\rangle = \langle Z, Y\rangle$ where $\langle\, ,\rangle$ denotes the scalar product of $\mathcal{H}$.

The unitary evolution describing the k-th interaction is given by U_k which acts non trivially like U on $\mathcal{H}_0 \otimes \mathcal{H}_k$ whereas it acts like the identity operator on the other copies. If ρ is a state on $\mathbf{\Gamma}$, the effect of the k-th interaction is:

$$\rho \rightarrow U_k\, \rho\, U_k^\star$$

Hence the result of the k first interactions is described by the operator V_k on $\mathcal{B}(\mathbf{\Gamma})$ defined by the recursive formula:

$$\left\{\begin{array}{ccc} V_{k+1} & = & U_{k+1}V_k \\ V_0 & = & I \end{array}\right. \tag{2}$$

and its effect on a state is given by:

$$\rho \rightarrow V_k\, \rho\, V_k^\star.$$

The above equation gives us the description of quantum repeated interaction. Let us move on to the principle of quantum repeated indirect measurement. The idea is to perform a measurement with respect of an observable of the field at each interaction.

Let $A = \sum_{j=1}^p \lambda_j P_j$ be any observable on $\mathcal{H}$, then we consider its natural ampliation which defines an observable on $\mathbf{\Gamma}$ by:

$$A^k := \bigotimes_{j=0}^{k} I \otimes \sum_{j=1}^{p} \lambda_j P_j \otimes \bigotimes_{j\geq k+1} I \tag{3}$$

As a consequence if ρ is any state on $\mathbf{\Gamma}$ the probabilistic theory of a quantum measurement with respect to A^k gives:

$$P[\text{to observe } \lambda_j] = Tr[\,\rho\, P_j^k\,], \quad j \in \{1, \dots, p\}$$

If we have observed the eigenvalue λ_j the "projection" postulate called "wave packet reduction" imposes the new state to be

$$\rho_j = \frac{P_j^k\, \rho\, P_j^k}{Tr[\,\rho\, P_j^k\,]}.$$

Remark: This new state is then the reference state of our system. If we want to perform another measurement of the observable A^k we obtain $P[\text{to observe } \lambda_j] = 1$. As a consequence a repeated interaction is necessary to obtain more than one information on $\mathcal{H}_0$.

This above description gives the principle of measurement on the k-th copy. The quantum repeated measurement principle is the combination of the measurement principle and the repeated quantum interactions. Physically it means that each photons interacts with the atom and we perform a measurement after each interaction. After each procedure we have a new state given by the projection postulate: this is our discrete quantum trajectory.

The initial state on $\mathbf{\Gamma}$ is chosen to be

$$\mu = \rho \otimes \bigotimes_{j\geq 1} \beta_j$$

where ρ is any state on $\mathcal{H}_0$ and each $\beta_i = \beta$ is the reference state on $\mathcal{H}$. We denote by μ_k the state representing the new state after the k first interactions, that is $\mu_k = V_k \mu V_k^\star$.

Let us now define the probabilistic framework and describe the effect of the successive measurements. We put $\Omega = \{1, \ldots, p\}$ and on $\Omega^{\mathbb{N}}$ we define the cylinders of size k:

$$\Lambda_{i_1,\ldots,i_k} = \{\omega \in \Omega^{\mathbb{N}} / \omega_1 = i_1, \ldots, \omega_k = i_k\}.$$

We endow $\Omega^{\mathbb{N}}$ with the σ-algebra generated by all these sets. This is the cylinder σ-algebra. Remarking that for all j, the unitary operator U_j commutes with all P^k for all $k < j$, for $\{i_1, \ldots, i_k\}$ (corresponding to the index of eigenvalues). Thus we can define the following non normalized state :

$$\begin{aligned} \tilde{\mu}(i_1, \ldots, i_k) &= I \otimes P_{i_1} \otimes \ldots \otimes P_{i_k} \otimes I \ldots \mu_k \, I \otimes P_{i_1} \otimes \ldots \otimes P_{i_k} \otimes I \ldots \\ &= P^k_{i_k} \ldots P^1_{i_1} \, \mu_k \, P^1_{i_1} \ldots P^k_{i_k}. \end{aligned}$$

So we can define a probability on $\Omega^{\mathbf{N}}$, defined on the cylinders:

$$P[\Lambda_{i_1,\ldots,i_k}] = Tr[\tilde{\mu}(i_1, \ldots, i_k)].$$

This probability satisfies the Kolmogorov consistency criterion, it defines then a probability on $\Omega^{\mathbf{N}}$. Hence we define the following random sequence of states:

$$\begin{array}{cccc} \tilde{\rho}^k(.) & \Omega^{\mathbf{N}} & \longrightarrow & \mathcal{B}(\mathbf{\Gamma}) \\ & \omega & \longmapsto & \tilde{\rho}_k(\omega_1 \ldots \omega_k) = \frac{\tilde{\mu}(\omega_1 \ldots \omega_k)}{Tr[\tilde{\mu}(\omega_1 \ldots \omega_k)]} \end{array}$$

This random sequence of states is our discrete quantum trajectory and the operator $\tilde{\rho}^k(i_1, \ldots, i_k)$ represents the state, if we have observed the results $(i_1, \ldots, i_k)$ during the k first measurement. This fact is precised in the following proposition.

Proposition 1 *Let $(\tilde{\rho}_k)$ be the above random sequence of states we have for all $\omega \in \Omega^{\mathbb{N}}$:*

$$\tilde{\rho}_{k+1}(\omega) = \frac{P^{k+1}_{\omega_{k+1}} U_{k+1} \, \tilde{\rho}_k(\omega) \, U^\star_{k+1} P^{k+1}_{\omega_{k+1}}}{Tr\left[\, \tilde{\rho}_k(\omega) \, U^\star_{k+1} P^{k+1}_{\omega_{k+1}} U_{k+1} \right]}.$$

This proposition is obvious but summarizes the quantum repeated measurement principle. The sequence $\tilde{\rho}_k$ is the quantum trajectory rendering the effect of the successive measurements on $\mathbf{\Gamma}$. The following theorem is an easy consequence of the previous proposition.

Theorem 1 *The sequence $(\tilde{\rho}^n)_n$ is a Markov chain valued on the set of states of $\mathcal{H}_0 \bigotimes_{i\geq 1} \mathcal{H}_i$ It is described as follows:*

$$P\left[\tilde{\rho}^{n+1} = \mu/\tilde{\rho}^n = \theta_n, \ldots, \tilde{\rho}^0 = \theta_0\right] = P\left[\tilde{\rho}^{n+1} = \mu/\tilde{\rho}^n = \theta_n\right]$$

If $\tilde{\rho}^n = \theta_n$ then $\tilde{\rho}^{n+1}$ takes one of the values:

$$\frac{P_i^{n+1}\,(U_{n+1}(\theta_n \otimes \beta)U_{n+1}^\star)\,P_i^{n+1}}{Tr\left[\,(U_{n+1}\,\theta_n\,U_{n+1}^\star)\,P_i^{n+1}\,\right]} \quad i = 1, \ldots, p$$

with probability $Tr\left[\,(U_{n+1}\,\theta_n\,U_{n+1}^\star)\,P_i^{n+1}\,\right]$.

In quantum theory it was assumed that we do not have access to the field (because it is more complicated), we just have access to the small system. So the mathematical tools rendering this phenomenon is the partial trace operation given by the following theorem.

Definition-Theorem 1 *If we have a state α on a tensor product $\mathcal{H} \otimes \mathcal{K}$. There exists a unique state η on $\mathcal{H}$ which is characterized by the property:*

$$\forall X \in \mathcal{B}(\mathcal{H}) \;\; Tr_{\mathcal{H}}[\,\eta\, X\,] = Tr_{\mathcal{H}\otimes\mathcal{K}}[\,\alpha(X \otimes I)\,].$$

Hence to obtain the trajectory concerning the small system we have to take the partial trace on $\mathcal{H}_0$. Let $\mathbf{E}_0$ denotes the partial trace on $\mathcal{H}_0$ with respect to $\bigotimes_{k\geq 1} \mathcal{H}_k$. We then define a random sequence of states on $\mathcal{H}_0$. For all ω in $\Omega^{\mathbb{N}}$ we put:

$$\rho_n(\omega) = \mathbf{E}_0[\tilde{\rho}_n(\omega)]. \tag{4}$$

This defines a sequence of state on $\mathcal{H}_0$ which contains the "partial" information given by the measurement and we have the following theorem which is a consequence of theorem (1).

Theorem 2 *The random sequence defined by formula* (4) *is a Markov chain with values in the set of states on $\mathcal{H}_0$. If $\rho_n = \chi_n$ then ρ_{n+1} takes one of the values:*

$$\mathbf{E}_0\left[\frac{I \otimes P_i\, U(\chi_n \otimes \beta)U^\star\, I \otimes P_i}{Tr[\,U(\chi_n \otimes \beta)U^\star\, I \otimes P_i]}\right] \;\; i = 1 \ldots p$$

with probability $Tr\,[U(\chi_n \otimes \beta)U^\star\, P_i]$.

Remark: Let us stress that $\frac{I\otimes P_i\, U(\chi_n\otimes\beta)U^\star\, I\otimes P_i}{Tr[\,U(\chi_n\otimes\beta)U^\star\, I\otimes P_i]}$ is a state on $\mathcal{H}_0 \otimes \mathcal{H}$, we have kept the notation $\mathbf{E}_0$ to denote the partial trace on $\mathcal{H}_0$.

The next section is devoted to the case $\mathcal{H}_0 = \mathcal{H} = \mathbf{C}^2$ which represents a two-level atom in contact with a photon stream. Because of physical consideration this particular case is often the central case in the literature concerning continuous measurement. The different results will be establish in this setting.

1.2 A two-level atom

In this section we want to establish a discrete quantum evolution equation for (ρ_n) which is a discrete equivalent of the Belavkin equation. In the previous section we have treated the general case, here we are going to work in 2 dimensionnal Hilbert spaces.

The main goal of this section is to obtain a formula of the following form:

$$\rho_{k+1} = f(\rho_k, X_{k+1}). \tag{5}$$

where $(X_k)_k$ is a sequence of random variables. In order to obtain such a formula we study how to obtain ρ_{k+1} through the measurement after the $(k+1)$-th interaction when the initial state after k procedures is ρ_k.

The state ρ_k can be namely considered as a initial state (according to the Markov property cf theorem (2)). Thus we consider a single interaction with a system $(\mathcal{H}, \beta)$ (actually this is the $k+1$-th copy). Remember that each Hilbert space are $\mathbf{C}^2$. We consider an observable of the form $A = \lambda_0 P_0 + \lambda_1 P_1$ and the unitary operator describing the interaction is a unitary 4×4 matrix. We consider it as an operator on $\mathcal{H}_0$:

$$U = \begin{pmatrix} L_{00} & L_{01} \\ L_{10} & L_{11} \end{pmatrix}$$

where each L_{ij} are operators on $\mathcal{H}_0$. In order to compute the state given by the projection postulate we choose a suitable basis. If $(X_0 = \Omega, X_1 = X)$ is an orthonormal basis of $\mathbf{C}^2$, for $\mathcal{H}_0 \otimes \mathcal{H}$ we consider the following basis $\Omega \otimes \Omega, X \otimes \Omega, \Omega \otimes X, X \otimes X$. This basis allows us to consider the above way of writing for U. For β we choose:

$$\beta = |\Omega\rangle\langle\Omega|$$

As a consequence, the state after the interaction is:

$$\mu_{k+1} = U(\rho_k \otimes \beta)U^\star = \begin{pmatrix} L_{00}\rho_k L_{00}^\star & L_{00}\rho_k L_{10}^\star \\ L_{10}\rho_k L_{00}^\star & L_{10}\rho_k L_{10}^\star \end{pmatrix}. \tag{6}$$

We apply the indirect quantum measurement principle. For the two possible results of the measurement we put:

$$\mathcal{L}_0(\rho_k) = \mathbf{E}_0[I \otimes P_0(\mu_{k+1}) I \otimes P_0] \tag{7}$$
$$\mathcal{L}_1(\rho_k) = \mathbf{E}_0[I \otimes P_1(\mu_{k+1}) I \otimes P_1]. \tag{8}$$

Thanks to the partial trace these are operators on $\mathcal{H}_0$. We denote the two probability by $p_{k+1} = Tr[\mathcal{L}_0(\rho_k)]$ and $q_{k+1} = Tr[\mathcal{L}_1(\rho_k)]$. The non normalized state: $\mathcal{L}_0(\rho_k)$ appears with probability p_{k+1} and $\mathcal{L}_1(\rho_k)$ with probability q_{k+1}.

Thanks to this two probabilities we can define a random variable ν_{k+1} on $\{0, 1\}$ by:

$$\begin{cases} \nu_{k+1}(0) = 0 & \text{with probability } p_{k+1} \\ \nu_{k+1}(1) = 1 & \text{with probability } q_{k+1} \end{cases}$$

As a consequence we can describe the state on $\mathcal{H}_0$ with the following equation. We have for all $\omega \in \Omega^{\mathbb{N}}$:

$$\rho_{k+1}(\omega) = \frac{\mathcal{L}_0(\rho_k(\omega))}{p_{k+1}(\omega)}(1 - \nu_{k+1}(\omega)) + \frac{\mathcal{L}_1(\rho_k(\omega))}{q_{k+1}(\omega)}\nu_{k+1}(\omega) \tag{9}$$

In order to obtain the final discrete quantum evolution equation we consider the centered and normalized random variable:

$$X_{k+1} = \frac{\nu_{k+1} - q_{k+1}}{\sqrt{q_{k+1}p_{k+1}}}.$$

We define the associated filtration on $\{0,1\}^{\mathbf{N}}$:

$$\mathcal{F}_k = \sigma(X_i, i \leq k).$$

So by construction we have $\mathbf{E}[X_{k+1}/\mathcal{F}_k] = 0$ and $\mathbf{E}[X_{k+1}^2/\mathcal{F}_k] = 1$. Thus we can write the discrete evolution equation for our quantum trajectory.

$$\rho_{k+1} = \mathcal{L}_0(\rho_k) + \mathcal{L}_1(\rho_k) + [-\sqrt{\frac{q_{k+1}}{p_{k+1}}}\mathcal{L}_0(\rho_k) + \sqrt{\frac{p_{k+1}}{q_{k+1}}}\mathcal{L}_1(\rho_k)]X_{k+1}. \tag{10}$$

The above equation can be considered in a general way and the unique solution starting from ρ_0 is our quantum trajectory. Here the time interaction is chosen arbitrarly to be one. In Section 3 we are going to consider this equation with a interaction time h which is supposed later to go to zero. From the physical point of view, we perform a sequence of measurements spaced by a time h. In section 3 we present the link between the different models and the convergence results. Before treating these questions we shall focus on the equations which are supposed to model the continuous measurement, this is the content of the next section.

1.3 Belavkin equation

It is commonly assumed that the evolution of a system undergoing a continuous measurement is described by stochastic differential equation. High technical tools like continuous quantum filtering are usually necessary to obtain rigorous results about existence and uniqueness. This heavy machinery is refered to fine properties of Von-Neumann algebra and needs important background. Otherwise heuristic but more intuitive rules can be used to obtain such description in a non rigorous way (see [4]).

The framework of our subject is the following. Consider a two-level atom (the small system) describing by $\mathbf{C}^2$ and any state ρ in interaction with an environment (classically described by a Fock space endowed with a reference state). The time evolution is given by a unitary-process (U_t) which satisfies a quantum Langevin equation (cf[13]). Without measurement the evolution of the small system is given by a norm continuous semigroup

$\{T_t\}_{t\geq 0}$ i.e $\rho_t = T_t(\rho)$. The Linblad generator of (T_t) is denoted by L and we have the master equation:

$$\frac{d\rho_t}{dt} = L(\rho_t) = -i[H, \rho_t] - \frac{1}{2}\{CC^\star, \rho_t\} + C\rho_t C^\star$$

where C is any operator and H is the Hamiltonian of the atom.

In the theory of time continuous measurement L is decomposed as the sum of $\mathcal{L} + \mathcal{J}$ where $\mathcal{J}$ represents the instantaneous state change taking place when detecting a photon, and $\mathcal{L}$ describes the smooth state variation in between these instants. These operators are defined by

$$\begin{aligned} \mathcal{L}(\rho) &= -i[H, \rho] - \frac{1}{2}\{CC^\star, \rho\} \\ \mathcal{J}(\rho) &= C\rho C^\star. \end{aligned}$$

Thanks to the works of Davies in [5] which describes the evolution of a state during a continuous measurement. We can obtain with more or less rigorous argument two different equations whose solutions are called quantum trajectories:

The diffusive equation (Homodyne detection experiment) is:

$$d\rho_t = L(\rho_t)dt + [\rho_t C^\star + C\rho_t - Tr\left(\rho_t(C + C^\star)\right)\rho_t]dW_t$$

where W_t design a one-dimensional brownian motion.

The jump equation (Resonnance fluorescence experiment) is:

$$d\rho_t = L(\rho_t)dt + \left[\frac{\mathcal{J}(\rho_t)}{Tr[\mathcal{J}(\rho_t)]} - \rho_t\right](d\tilde{N}_t - Tr[\mathcal{J}(\rho_t)]dt)$$

where $\tilde{N}_t$ is assumed to be a counting process with stochastic intensity $\int_0^t Tr[\mathcal{J}(\rho_s)]ds$.

The main goal of this article is the justification of the Belavkin model through the previous discrete description. In the last section the continuous equation appears as a limit of the discrete equation. Such convergence theorem need a deep study of the continuous stochastic differential equation. For instance we don't speak about validity, we are going to study such equations in a general way.

In this paper we consider only the diffusive case, the jump-equation and all convergence theorems refering to this case are treated in details in [14] with different techniques.

Let ρ_0 be any state, the aim is to show existence and uniqueness for the stochastic differential equation:

$$\rho_t = \rho_0 + \int_0^t L(\rho_s)ds + \int_0^t [\rho_s C^\star + C\rho_s - Tr[(\rho_s(C + C^\star))\rho_s]dW_s. \tag{11}$$

Classical theorems concerning such equations can not be applied because the coefficients are not Lipschitz. A finer study is necessary to come to the result. From a mathematical-physical point of view if there exists a solution we must check that the solution-process is valued on set of states. The problem of existence and the fact that the solution must be a state valued process are actually not independent. Before to answer the general problem we investigate the question of purity.

An important feature of the differential equation is that it preserves the property to be a pure state. In quantum theory a pure state is a one dimensionnal projector. Indeed if the initial state is pure and if there is a solution, the solution-process is valued on the set of pure state. This idea is mentioned in the following proposition and will be resumed in the final theorem:

Proposition 2 *Let (W_t) be a standard Brownian motion on $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ and let $|\psi_0\rangle$ be any vector in $\mathbf{C}^2$ of norm one. Let $\nu_t = \frac{1}{2}\langle \psi_t, (C + C^\star)\psi_t\rangle$, if the following stochastic equation:*

$$d|\psi_t\rangle \;=\; (C - \nu_t I)|\psi_t\rangle dW_t + \left(-iH - \frac{1}{2}\left(C^\star C - 2\nu_t C + \nu_t^2 I\right)\right)|\psi_t\rangle dt \qquad (12)$$

admits a solution $(|\psi_t\rangle)$ then almost surely for all t $\|\psi_t\| = 1$ and the process $(|\psi_t\rangle\langle\psi_t|)$ is a solution of the diffusive Belavkin equation (11).

Proof: Let $|\psi_0\rangle$ be any vector in $\mathbf{C}^2$ and let $(|\psi_t\rangle)$ be the solution of (12). We can compute $d\|\psi_t\|^2$. Thanks to Ito formulas: $dW_t dW_t = dt$ and $dW_t dt = 0$ and the fact that H is self-adjoint a straightforward computation shows that:

$$\begin{aligned}
d\|\psi_t\|^2 &= d\langle \psi_t, \psi_t\rangle = \langle d\psi_t, \psi_t\rangle + \langle \psi_t, d\psi_t\rangle + \langle d\psi_t, d\psi_t\rangle \\
&= \langle (C - \nu_t I)\psi_t, \psi_t\rangle dW_t + \langle (-iH - \frac{1}{2}(C^\star C - 2\nu_t C + \nu_t^2 I)\psi_t, \psi_t\rangle dt \\
&\quad + \langle \psi_t, (C - \nu_t I)\psi_t\rangle dW_t + \langle \psi_t, (-iH - \frac{1}{2}(C^\star C - 2\nu_t C + \nu_t^2 I)\psi_t\rangle dt \\
&\quad + \langle (C - \nu_t I)\psi_t, (C - \nu_t I)\psi_t\rangle dt \\
&= (2\nu_t - 2\nu_t\langle \psi_t, \psi_t\rangle) dW_t
\end{aligned}$$

If $\|\psi_0\|^2 = 1$ it implies that almost surely for all $t > 0$:

$$\|\psi_t\|^2 = \|\psi_0\|^2 = 1.$$

We define for all $t \geq 0$ $\rho_t = |\psi_t\rangle\langle\psi_t|$, thanks to the fact that $\|\psi_t\| = 1$ we then have for all $y \in \mathbf{C}^2$:

$$\rho_t|y\rangle = \langle \psi_t, y\rangle |\psi_t\rangle.$$

So we can compute $d\,\rho_t|y\rangle$ by the Ito formula:

$$\begin{aligned}
& d\,\rho_t|y\rangle = \langle d\psi_t, y\rangle|\psi_t\rangle + \langle\psi_t, y\rangle d|\psi_t\rangle + \langle d\psi_t, y\rangle d|\psi_t\rangle \\
= \; & \langle (C-\nu_t)\psi_t, y\rangle|\psi_t\rangle dW_t + \langle(-iH - \frac{1}{2}(C^\star C - 2\nu_t C + \nu_t^2)\psi_t, y\rangle|\psi_t\rangle dt \\
& + \langle\psi_t, y\rangle(C-\nu_t)|\psi_t\rangle dW_t + \langle\psi_t, y\rangle(-iH - \frac{1}{2}(C^\star C - 2\nu_t C + \nu_t^2)|\psi_t\rangle dt \\
& + \langle (C-\nu_t)\psi_t, y\rangle(C-\nu_t)|\psi_t\rangle\, dt
\end{aligned}$$

Let us recognize the equation (11). It was clear that $\nu_t = \frac{1}{2}Tr[|\psi_t\rangle\langle\psi_t|(C+C^\star)]$. As a consequence the term in front of the Brownian motion becomes:

$$\begin{aligned}
& \langle (C-\nu_t)\psi_t, y\rangle|\psi_t\rangle + \langle\psi_t, y\rangle(C-\nu_t)|\psi_t\rangle \\
= \; & (C|\psi_t\rangle\langle\psi_t| + |\psi_t\rangle\langle\psi_t|C^\star - Tr[|\psi_t\rangle\langle\psi_t|(C+C^\star)]|\psi_t\rangle\langle\psi_t|)\,|y\rangle
\end{aligned}$$

A similar computation show that the term in front of dt is:

$$L(|\psi_t\rangle\langle\psi_t|)|y\rangle$$

Hence we recognize the Belavkin equation (9) and the proposition is proved. □

We can formulate the theorem concerning the existence and the uniqueness of a solution of (11):

Theorem 3 *Let $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ be a probabilistic space which supports a standard Brownian motion (W_t) and let ρ_0 be any state, the stochastic differential equation:*

$$\rho_t = \rho_0 + \int_0^t L(\rho_s)ds + \int_0^t [\rho_s C^\star + C\rho_s - Tr[(\rho_s(C+C^\star))\,\rho_s]dW_s$$

admits a unique solution (ρ_t) valued on the set of states and defined for all $t \in [0, \infty[$.

Furthemore if the initial condition is a pure state the solution takes value in the set of pure states. The stochastic differential equation for a wave function is given by:

$$d|\psi_t\rangle \;=\; (C-\nu_t)|\psi_t\rangle dW_t + \left(-iH - \frac{1}{2}\left(C^\star C - 2\nu_t C + \nu_t^2\right)\right)|\psi_t\rangle dt$$

where $\nu_t = \frac{1}{2}\langle\psi_t, (C+C^\star)\psi_t\rangle$.

Proof: As the coefficients are not Lipschitz we can not apply directly the usual theorem for SDE's. However the coefficients are C^∞, so locally Lipschitzian and we can use a truncature method. Our equation is of the following form :

$$d\rho_t = L(\rho_t)dt + \Theta(\rho_t)dW_t \tag{13}$$

where Θ is C^∞ and $\Theta(A) = AC^\star + CA - Tr\left[A(C + C^\star)\right] A$. Let $k \in \mathbb{N}$, $k > 1$, we define the truncation function φ_k from $\mathbb{R}$ to $\mathbb{R}$ defined by

$$\varphi_k(x) = \begin{cases} -k & \text{if } x \leq -k \\ x & \text{if } -k \leq x \leq k \\ k & \text{if } -k \leq x \leq k \end{cases}$$

For a matrix $A = (a_{ij})$ we define by extension $\tilde{\varphi}_k(A) = (\varphi_k(Re(a_{ij})) + i\varphi_k(Im(a_{ij})))$. Thus $\Theta \circ \tilde{\varphi}_k$ is a Lipschitz function. Now we consider the truncated equation:

$$d\rho_{k,t} = L \circ \tilde{\varphi}_k(\rho_{k,t})dt + \Theta \circ \tilde{\varphi}_k(\rho_{k,t})dW_t.$$

The theorem concerning stochastic differential equation driven by a drift term and a Brownian motion can be applied and there exists a unique solution $t \mapsto \rho_{k,t}$ defined for all t. Besides the solution is continuous.

We define the stopping random time

$$T_k = \inf\{t, \exists (ij)/|Re(a_{ij}(\rho_{k,t}))| = k \, or \, |Im(a_{ij}(\rho_{k,t}))| = k\}.$$

As ρ_0 is a state, for a suitable norm we have $\|\rho_0\| \leq 1$. Thanks to continuity, if k is choosed large enough we have $T_k > 0$ and for all $t \leq T_k$ we have $\tilde{\varphi}_k(\rho_{k,t}) = \rho_{k,t}$. Thus $t \mapsto \rho_{k,t}$ is the unique solution of equation (11) (whithout truncation) on $[0, T_k]$. The classical method to solve an equation with non Lipschitz coefficients is to put $T = \lim_k T_k$ and to show that $T = \infty$.

In addition to the proof of existence of a solution we must prove that the process is valued on set of states. So when we deal with any state ν we have $\|\nu\| \leq 1$ and we have $|\nu(ij)| \leq 1$. Hence if we prove that on $[0, T_2]$, the process $(\rho_{2,t})$ is valued on set of states it proves that $T_2 = \infty$ a.s and then we have proved that there exists a unique solution valued on set of states. Let us prove this fact.

In the proof of the existence and uniqueness of a solution in the case of Cauchy-Lipschitz coefficients we create a sequence which converges to the solution:

$$\begin{cases} \rho_{n+1}(t) &= \rho_n(0) + \int_0^t L \circ \tilde{\varphi}_k(\rho_n(s))ds + \int_0^t \Theta \circ \tilde{\varphi}_k(\rho_n(s))dW_s \\ \rho_0(t) &= \rho \end{cases} \tag{14}$$

With the right definition of Θ and L if ρ_0 is a state it is easy to see that this sequence is self-adjoint with trace one. This conditions are closed and at the limit the process is self-adjoint with trace one. The condition of positivity does not result of this sequence.

We introduce the random time:

$$T^0 = \inf\{t \leq T_2 / \exists X \in \mathbf{C}^2 / \langle X, \rho_{2,t} X\rangle = 0\} \tag{15}$$

We have $\langle X, \rho_0 X\rangle \geq 0$ for all X, so by continuity we have $\langle X, \rho_{2,t} X\rangle \geq 0$ on $[0, T^0]$ which implies that $\rho_{2,t}$ is a state for all $t \leq T^0$.

If $T^0 = T_2$ a.s the result is proved. Otherwise we have $T^0 < T_2$, hence by continuity there exists X such that $\langle X, \rho_{2,T^0} X\rangle = 0$ and for all Y $\langle Y, \rho_{2,T^0} Y\rangle \geq 0$. It implies that

ρ_{2,T^0} is a pure state because we work in dimension 2. Let us denote by ψ_{T^0} a vector of norm one such that $\rho_{2,T^0} = |\psi_{T^0}\rangle\langle\psi_{T^0}|$. We consider the equation:

$$d|\psi_t\rangle \ = \ (C-\nu_t)|\psi_t\rangle dW_t + \left(-iH - \frac{1}{2}\left(C^\star C - 2\nu_t C + \nu_t^2\right)\right)|\psi_t\rangle dt$$

with ψ_{T^0} as initial condition. The problem of existence and uniqueness for this equation is solved by a truncation method too. The fact that if we have a solution it is of norm one 1 shows that the solution obtained by truncation (defined for all t) is actually the solution of (12). The proposition (2) and the uniqueness of $\rho_{2,t}$ on $[T^0, T_2]$ shows that the solution:

$$|\psi_t\rangle = |\psi_{T^0}\rangle + \int_{T^0}^t (C-\nu_s)|\psi_s\rangle dW_s + \left(-iH - \frac{1}{2}\left(C^\star C - 2\nu_s C + \nu_s^2\right)\right)|\psi_s\rangle ds$$

defines a process $(|\psi_t\rangle\langle\psi_t|)$ equals to $\rho_{2,t}$ on $[T^0, T_2]$. Hence the process obtained by truncation is valued on set of states and the result is proved. $\square$

If the initial condition is a state the Belavkin equation (11) admits a unique solution valued on set of states. Moreover if there exist t_0 such that ρ_{t_0} is a pure state, for all $t \geq t_0$, the state ρ_t is pure too.

1.4 Change of measure

Usually the stochastic equation appearing in the litterature is of the following form:

$$\rho_t = \rho_0 + \int_0^t L(\rho_s)ds + \int_0^t \left[\rho_s C^\star + C\rho_s - Tr[\rho_s(C+C^\star)]\right] d\tilde{W}_s \tag{16}$$

where:

$$\tilde{W}_t = W_t - \int_0^t Tr[\rho_s(C+C^\star)]ds \tag{17}$$

In the last section we will show a convergence of a discrete process to the solution of equation (11), the link between the two different equation is given by the Girsanov's theorem (see [16] for a good introduction):

Theorem 4 *Let (W_t) be a standard brownian motion on $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ and let H be a càdlàg process. Let:*

$$X_t = \int_0^t H_s ds + W_t \tag{18}$$

and define a new probability by $\frac{dQ}{dP} = \exp\left(-\int_0^T H_s dWs - \frac{1}{2}\int_0^T H_s^2 ds\right)$ for some $T > 0$. Hence under Q, the process (X_t) is a standard brownian motion for $0 \leq t \leq T$.

The link between the two equation is then obvious. Let (ρ_t) be the solution of equation (11) given by the theorem (3) on $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$. For some $T > 0$ we define the probability Q by:

$$\frac{dQ}{dt} = \exp\left(\int_0^T Tr[\rho_t(C + C^\star)]dW_s - \frac{1}{2}\int_0^T Tr[\rho_t(C + C^\star)]^2 ds\right) \tag{19}$$

The above theorem claim that $\tilde{W}_t$ is the a standard Brownian motion under Q for $0 \leq t \leq T$. As a consequence in the next section (concerning approximations and convergence theorems), we are going to consider (W_t) when we deal with a standard Brownian motion but we keep in mind that it is equivalent with the process which appears in physical literature.

2 Approximation and convergence

In this section we will construct a discrete quantum trajectory which converges to the solution of the diffusive Belavkin equation. The guiding idea is the fact that the discrete quantum trajectory satisfies a discrete stochastic equation which is of the same form as the continuous one.

2.1 The discrete approximation

At the beginning of the section 2 we have announced that there are essentially two kinds of Belavkin equation. We are going to see how the diffusive case appears from discrete process.

In section 1 we have obtained the following equation:

$$\rho^{k+1} = \mathcal{L}_0(\rho^k) + \mathcal{L}_1(\rho^k) + \left[-\sqrt{\frac{q_{k+1}}{p_{k+1}}}\mathcal{L}_0(\rho^k) + \sqrt{\frac{p_{k+1}}{q_{k+1}}}\mathcal{L}_1(\rho^k)\right] X_{k+1} \tag{20}$$

Hence we have:

$$\begin{aligned}
\rho^{k+1} - \rho^0 &= \sum_{i=0}^{k}[\rho^{i+1} - \rho^i] \\
&= \sum_{i=0}^{k}[\mathcal{L}_0(\rho^i)) + \mathcal{L}_1(\rho^i) - \rho^i] \\
&\quad + \sum_{i=0}^{k}\left[-\sqrt{\frac{q_{i+1}}{p_{i+1}}}\mathcal{L}_0(\rho^i) + \sqrt{\frac{p_{i+1}}{q_{i+1}}}\mathcal{L}_1(\rho^i)\right] X_{i+1} \qquad (21)
\end{aligned}$$

The discrete process ρ^k appears as the solution of a kind of a discrete time stochastic differential equation. This idea is going to be developed in order to obtain an approximation of the solution of (11).

We want now to introduce a discretisation of time. Consider a partition of $[0,T]$ in subintervals of equal size $\frac{1}{n}$. The time of interaction is supposed now to be $\frac{1}{n}$, the dynamic laws concerning the evolution of an open quantum system imposed that the unitary operator of evolution depends on the time interaction. We then have:

$$U(n) = \begin{pmatrix} L_{00}(n) & L_{01}(n) \\ L_{10}(n) & L_{11}(n) \end{pmatrix}.$$

The work of Attal-Pautrat [3] has shown that the asymptotics of the coefficients $L_{ij}(n)$ must be properly rescaled in order to obtain a non-trivial limit. Indeed they have shown that $V_{[nt]} = U_{[nt]}(n)\dots U_1(n)$ which represents the discrete dynamic of quantum repeated interaction converges to an operator V_t representing the continuous dynamic. Through the measurement theory we find a part of this result again. We choose:

$$L_{00}(n) = I + \frac{1}{n}\left(-iH - \frac{1}{2}CC^\star\right) + \circ\left(\frac{1}{n}\right) \tag{22}$$

$$L_{10}(n) = \frac{1}{\sqrt{n}}C + \circ\left(\frac{1}{n}\right) \tag{23}$$

The corresponding Hamiltonian $H_{tot}(n)$ describing the interaction during a time interval $\frac{1}{n}$ on $\mathcal{H}_0 \otimes \mathcal{H}$ is of the following form:

$$H_{tot}(n) = H \otimes I + I \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \frac{1}{\sqrt{n}}\left[C \otimes \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + C^\star \otimes \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right] + \circ\left(\frac{1}{n}\right)$$

where H is the Hamiltonian of the small system and C is any operator.

With the time discretization we obtain:

$$\begin{aligned} \rho^{k+1}(n) &= \mathcal{L}_0(n)(\rho^k(n)) + \mathcal{L}_1(n)(\rho^k(n)) \\ &+ \left[-\sqrt{\frac{q_{k+1}(n)}{p_{k+1}(n)}}\mathcal{L}_0(n)(\rho^k(n)) + \sqrt{\frac{p_{k+1}(n)}{q_{k+1}(n)}}\mathcal{L}_1(n)(\rho^k(n))\right] X_{k+1}(n) \end{aligned}$$

The sequence of random variables $(X_k(n))$ is defined through the two probabilities:

$$\begin{aligned} p_{k+1} &= Tr[\mathcal{L}_0(\rho^k)] \\ q_{k+1} &= Tr[\mathcal{L}_1(\rho^k)]. \end{aligned}$$

Each $\mathcal{L}_i$ depends on the measured observable: $A = \lambda_0 P_0 + \lambda_1 P_1$.

The aim of this section is to show the convergence of $\rho^{[nt]}(n)$ to the solution of diffusive Belavkin equation (11). The Brownian noise will appear thanks to the sequence (X_k). By definition we have:

$$X_k(n)(i) = \begin{cases} -\sqrt{\frac{q_{k+1}(n)}{p_{k+1}(n)}} & \text{with probability} \quad p_{k+1}(n) \text{ if } i = 0 \\ \sqrt{\frac{p_{k+1}(n)}{q_{k+1}(n)}} & \text{with probability} \quad q_{k+1}(n) \text{ if } i = 1 \end{cases} \tag{24}$$

As the probability and the operators $\mathcal{L}_i$ depends on the observable, we are going to classify the observables in order to determine which ones give the diffusive nature.

If the observable is of the form $A = \lambda_0 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \lambda_1 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, we obtain the asymptotics for the probabilities:

$$\begin{aligned} p_{k+1}(n) &= 1 - \frac{1}{n} Tr\left[\mathcal{J}(\rho^k(n))\right] + \circ\left(\frac{1}{n}\right) \\ q_{k+1}(n) &= \frac{1}{n} Tr\left[\mathcal{J}(\rho^k(n))\right] + \circ\left(\frac{1}{n}\right) \end{aligned}$$

The discrete equation becomes:

$$\begin{aligned} \rho^{k+1}(n) - \rho^k(n) &= \frac{1}{n} L(\rho^k(n)) + \circ(\frac{1}{n}) \\ &+ \left[\frac{\mathcal{J}(\rho^k(n))}{Tr\left[\mathcal{J}(\rho^k(n))\right)]} - \rho^k(n) + \circ(1)\right] \sqrt{q_{k+1}(n) p_{k+1}(n)}\, X_{k+1}(n) \end{aligned}$$

If the observable is non diagonal in the basis (Ω, X), we consider $P_0 = \begin{pmatrix} p_{00} & p_{01} \\ p_{10} & p_{11} \end{pmatrix}$ and $P_1 = \begin{pmatrix} q_{00} & q_{01} \\ q_{10} & q_{11} \end{pmatrix}$ we have:

$$\begin{aligned} p_{k+1} &= p_{00} + \frac{1}{\sqrt{n}} Tr\left[\rho^k (p_{01} C + p_{10} C^\star)\right] + \frac{1}{n} Tr\left[\rho^k p_{00} (C + C^\star)\right] + \circ\left(\frac{1}{n}\right) \\ q_{k+1} &= q_{00} + \frac{1}{\sqrt{n}} Tr\left[\rho^k (q_{01} C + q_{10} C^\star)\right] + \frac{1}{n} Tr\left[\rho^k q_{00} (C + C^\star)\right] + \circ\left(\frac{1}{n}\right) \end{aligned}$$

The discrete equation becomes then

$$\begin{aligned} \rho^{k+1} - \rho^k &= \frac{1}{n} L(\rho^k) + \circ\left(\frac{1}{n}\right) + \left[e^{i\theta} C \rho^k + e^{-i\theta} \rho^k C^\star\right. \\ &\left. - Tr[\rho^k (e^{i\theta} C + e^{-i\theta} C^\star)]\, \rho^k + \circ(1)\right] \frac{1}{\sqrt{n}} X_{k+1} \end{aligned}$$

There appears a deterministic parameter θ which represents a kind of phase. This real parameter can be express thanks to the coefficients of the projector P_i and can be introduced in the continuous equation. The results are the same. We then consider $\theta = 0$ for the rest of the paper.

In [14] it is shown that the first case (when A is diagonal) gives the jump-Belavkin equation. In this section we shall show that the second case give rise to the diffusive Belavkin equation.

2.2 Convergence theorems

Before presenting the theorem concerning the convergence of the quantum trajectories, we expose a theorem concerning the average of the processes.

Theorem 5 *Let (Ω, X) be any orthonormal basis of $\mathbb{C}^2$. For all non diagonal observable A, the deterministic function $t \to \mathbf{E}[\rho^{[nt]}(n)]$ converges in $L^\infty([0,T])$ to the function $t \to \mathbf{E}[\rho_t]$ when n goes to infinity. That is:*

$$\sup_{0<s<T} \|\mathbf{E}[\rho^{[ns]}(n)] - \mathbf{E}[\rho_s]\| \underset{n\to\infty}{\longrightarrow} 0.$$

Furthemore the function $t \to \mathbf{E}[\rho_t]$ is the solution of the master equation:

$$d\,\nu_t = L(\nu_t)dt.$$

Proof: First of all we show the second part of theorem. We can consider the function $t \to \mathbf{E}[\rho_t]$ because we have existence and uniqueness of the solution which is integrable (because ρ_t is a state for all t). It was obvious that this deterministic function is process valued on set of states. As ρ_0 is a deterministic state we must show:

$$\mathbf{E}[\rho_t] = \rho_0 + \int_0^t L(\mathbf{E}[\rho_s])ds \tag{25}$$

The state valued process (ρ_t) satisfies:

$$\rho_t = \rho_0 + \int_0^t L(\rho_s)ds + \int_0^t [\rho_s C^\star + C\rho_s - Tr\left(\rho_s(C + C^\star)\right)\rho_s]dW_s$$

According to the fact that the process (W_t) is a martingale and that the process (ρ_t) is predictable because continuous, the properties of stochastic integral with respect to a martingale give:

$$\mathbf{E}\left[\int_0^t [\rho_s C^\star + C\rho_s - Tr\left(\rho_s(C + C^\star)\right)\rho_s]dW_s\right] = 0.$$

Hence we have by linearity of L:

$$\begin{aligned}
\mathbf{E}[\rho_t] &= \rho_0 + \int_0^t \mathbf{E}[L(\rho_s)]ds \\
&= \rho_0 + \int_0^t L(\mathbf{E}[\rho_s])ds
\end{aligned}$$

We then have the integral form of the solution of the master equation and the second part is proved.

We shall now compare $\mathbf{E}[\rho^{[nt]}(n)]$ with $\mathbf{E}[\rho_t]$ in order to obtain the convergence result. Like in the continuous case, the martingale argument is replaced by the fact that the process (X_k) is centered. Remember that we have:

$$\mathbf{E}[X_{k+1}] = \mathbf{E}[\mathbf{E}[X_{k+1}/\mathcal{F}_k]] = 0$$

As a consequence, considering $k = [nt]$ and taking expectation in the discrete equation we have:

$$\mathbf{E}[\rho^{[nt]}(n)] - \rho^0 = \sum_{i=0}^{[nt]-1} \frac{1}{n} L(\mathbf{E}[\rho^k(n)] + \circ(\frac{1}{n})$$

This is a kind of Euler scheme and we can conclude with a discrete Gronwall lemma to have the convergence:

$$\sup_{0<s<t} \|\mathbf{E}[\rho^{[ns]}(n)] - \mathbf{E}[\rho_s]\| \underset{n\to\infty}{\longrightarrow} 0$$

□

The average of the discrete process is then an approximation of the average of ρ_t. In [3] this result was shown in the case of repeated interaction whithout measurement, it is a consequence of the asymptotics of the unitary-operator coefficients, so it justifies the choice of the coefficients of $U(n)$. We are going to prove a similar result for the processes.

The discrete process which is the candidate to converge to the diffusive quantum trajectory satisfies for $k = [nt]$

$$\rho^{[nt]} - \rho^0 \quad = \quad \sum_{i=0}^{[nt]-1} \frac{1}{n} L(\rho^k(n)) + \circ\left(\frac{1}{n}\right) + \sum_{i=0}^{[nt]-1} [\Theta(\rho^k) + \circ(1)] \frac{1}{\sqrt{n}} X_{i+1}$$

Thanks to this equation we can define the processes:

$$\begin{aligned}
W_n(t) &= \frac{1}{\sqrt{n}} \sum_{k=1}^{[nt]} X_k(n) \\
V_n(t) &= \frac{[nt]}{n} \\
\rho_n(t) &= \rho^{[nt]}(n) \\
\varepsilon_n(t) &= \sum_{i=0}^{[nt]-1} \circ(\frac{1}{n}) + \sum_{i=0}^{[nt]-1} \circ(1) \frac{1}{\sqrt{n}} X_{i+1}
\end{aligned}$$

By observing that this four processes are piecewise constant we can write the process $(\rho_n(t))_{t\geq 0}$ like a solution of a stochastic differential equation in the following way:

$$\rho_n(t) \quad = \quad \rho_0 + \varepsilon_n(t) + \int_0^t L(\rho_n(s-)) dV_n(s) + \int_0^t \Theta(\rho_n(s-)) dW_n(s)$$

We will use a theorem of Kurtz and Protter (cf [12]) to prove the convergence. If we consider a process define by:

$$X_n(t) = \rho_0 + \varepsilon_n(t) + \int_0^t L(X_n(s-)dV_n(s) + \int_0^t \Theta(X_n(s-))dW_n(s)$$

Theorem 6 *Suppose that W_n is a martingale and V_n is a finite variation process. Assume that for each $t \geq 0$:*

$$\sup_n \mathbf{E}[[W_n, W_n]_t] < \infty$$
$$\sup_n \mathbf{E}[T_t(V_n)] < \infty$$

and that $(W_n, V_n, \varepsilon_n)$ converges in distribution to $(W, V, 0)$ where W is a standard brownian motion and $V(t) = t$ for all t.

Suppose that X satisfies:

$$X_t = X_0 + \int_0^t L(X_s)ds + \int_0^t \Theta(X_s)dW_s$$

and that the solution of this stochastic differential equation is unique. Then X_n converges in distribution to X.

Recall that $[X, X]$ is defined for a semi-martingale by the formula $[X, X]_t = X_t^2 - \int_0^t X_{s-}dX_s$. The operation T_t denotes the total variation of a finite variation process. For more details see [16].

The convergence in distribution means the convergence in distribution for stochastic processes. We need another theorem to prove this type of convergence:

Theorem 7 *Let (M_n) be a sequence of martingales. Suppose that*

$$\lim_{n\to\infty} \mathbf{E}[\sup_{s\leq t} |M_n(s) - M_n(s-)|] = 0$$

and

$$[M_n, M_n]_t \underset{n\to\infty}{\longrightarrow} t.$$

Then M_n converges in distribution to a standard Brownian motion. The conclusion is the same if we have:

$$\lim_{n\to\infty} \mathbf{E}\left[|[M_n, M_n]_t - t|\right] = 0.$$

Let us verify the different hypothesese to apply these theorems. We define a filtration for the process $(W_n(.))$:

$$\mathcal{F}_t^n = \sigma(X_i, i \leq [nt]).$$

Proposition 3 *We have that $(W_n(.), \mathcal{F}^n_.)$ is a martingale. The process $(W_n(.))$ converges to a standard Brownian motion $W_.$ when n goes to infinity and $sup_n \mathbf{E}[[W_n, W_n]_t] < \infty$.*

Furthemore, we have the convergence in distribution for the process $(W_n, V_n, \varepsilon_n)$ to $(W, V, 0)$ when n goes to infinity.

Proof: Thanks to the definition of the random variable X_k, we have $\mathbf{E}[X_{i+1}/\mathcal{F}^n_i] = 0$ which implies $\mathbf{E}[\frac{1}{n}\sum_{i=[ns]+1}^{[nt]} X_i/\mathcal{F}^n_s] = 0$ for $t > s$. Thus if $t > s$ we have the martingale property:

$$\mathbf{E}[W_n(t)/\mathcal{F}^n_s] = W_n(s) + \mathbf{E}\left[\frac{1}{\sqrt{n}}\sum_{i=[ns]+1}^{[nt]} X_i/\mathcal{F}^n_s\right] = W_n(s).$$

By definition of $[Y, Y]$ for a stochastic process we have

$$[W_n, W_n]_t = W_n(t)^2 - 2\int_0^t W_n(s-)dW_n(s) = \frac{1}{n}\sum_{i=1}^{[nt]} X_i^2$$

Thus we have

$$\begin{aligned}\mathbf{E}[[W_n, W_n]_t] &= \frac{1}{n}\sum_{i=1}^{[nt]}\mathbf{E}[X_i^2] = \frac{1}{n}\sum_{i=1}^{[nt]}\mathbf{E}[\mathbf{E}[X_i^2/\sigma\{X_l, l < i\}]] \\ &= \frac{1}{n}\sum_{i=1}^{[nt]} 1 = \frac{[nt]}{n}.\end{aligned}$$

Hence we have $\sup_n \mathbf{E}[[W_n, W_n]_t] \leq t < \infty$.

Let us prove the convergence of (W_n) to $(W_.)$. According to the theorem (7) we must prove that:

$$\lim_{n\to\infty}\mathbf{E}[|[M_n, M_n]_t - t|] = 0$$

It is a convergence in L_1. We are going to prove a convergence in L_2. In order to show the convergence in L_2 we will use the following property: $\mathbf{E}[X_i^2] = \mathbf{E}[\mathbf{E}[X_i^2/\sigma\{X_l, l < i\}]] = 1$ and if $i < j$ $\mathbf{E}[(X_i^2-1)(X_j^2-1)] = \mathbf{E}[(X_i^2-1)(X_j^2-1)/\sigma\{X_l, l < j\}]] = \mathbf{E}[(X_i^2-1)]\mathbf{E}[(X_j^2-1)] = 0$. Thus we have:

$$\begin{aligned}\mathbf{E}[([W_n, W_n]_t - \frac{[nt]}{n})^2] &= \frac{1}{n^2}\sum_{i=1}^{[nt]}\mathbf{E}[(X_i^2 - 1)^2] + \frac{1}{n^2}\sum_{i<j}\mathbf{E}[(X_i^2 - 1)(X_j^2 - 1)] \\ &= \frac{1}{n^2}\sum_{i=1}^{[nt]}\mathbf{E}[(X_i^2 - 1)^2]\end{aligned}$$

Thanks to the fact that p_{00} and q_{00} are not equal to zero (because the observable is not diagonal) $\mathbf{E}[(X_i^2 - 1)^2]$ is bounded uniformly in i so we have:

$$\lim_{n\to\infty}\mathbf{E}[([W_n, W_n]_t - \frac{[nt]}{n})^2] = 0$$

Like $\frac{[nt]}{n} \longrightarrow t$ in L_2 we have the desired convergence. The convergence in distribution of (W_n) and (V_n) implies the convergence of (ε_n) to 0. $\square$

This property is the essential point in the theorem of Kurtz an Protter. Thus we can express the final theorem.

Theorem 8 *Let (Ω, X) be any orthonormal basis of $\mathbb{C}^2$ and A be any observable non diagonal (in this basis). Let ρ be any state on $\mathbb{C}^2$.*

Let $(\rho^{[nt]}(n))$ be the discrete quantum trajectory obtained from the quantum repeated measurement principle with respect to A. The process $(\rho^{[nt]}(n))$ then satisfies:

$$\rho^{[nt]}(n) = \rho_0 + \sum_{i=0}^{[nt]-1} \frac{1}{n} L(\rho^k(n)) + \circ\left(\frac{1}{n}\right) + \sum_{i=0}^{[nt]-1} [\Theta(\rho^k) + \circ(1)] \frac{1}{\sqrt{n}} X_{i+1}$$

Let (ρ_t) be the solution of the diffusive Belavkin equation (11) *which satisfies:*

$$\rho_t = \rho_0 + \int_0^t L(\rho_s) ds + \int_0^t \Theta(\rho_s) dW_s.$$

Thus we have the following convergence in distribution for stochastic process:

$$(\rho^{[nt]}(n)) \xrightarrow[n \to \infty]{\mathcal{D}} (\rho_t)$$

Proof: It is a simply compilation of the two theorems and the property for $(W_n(.))$. To conclude we use the existence and uniqueness property proved in the second section (see theorem (3)). $\square$

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